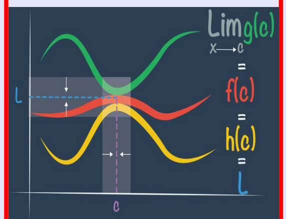
Math 261

Fall 2022

Lecture 14

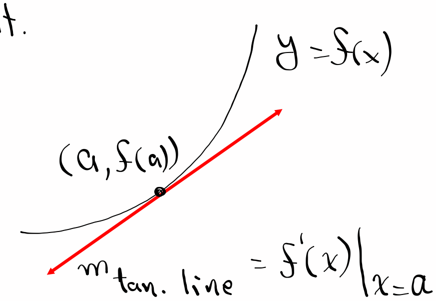


Given a function $f(x)$, the first derivative of $f(x)$, $f'(x)$ F-Prime of X is given by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists

$f'(x)$ is a formula for the slope of the tangent line to the graph of $f(x)$ at any point.



Find $f'(x)$ for $f(x)=x^3+3x$

By Def. of $f'(x)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(x+h)^3+3(x+h)-x^3-3x}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3+\cancel{3x}+3h-\cancel{x^3}-\cancel{3x}}{h}$$

$$=\lim_{h\to 0}\frac{\cancel{h}(3x^2+3xh+h^2+3)}{\cancel{h}}=\lim_{h\to 0}(3x^2+\underbrace{3xh}_{0}+\underbrace{h^2}_{0}+3)$$

$$=3x^2+3$$

$f(x)=x^3+3x$

$f'(x)=3x^2+3$

$=3(x^2+1)$

$f'(x)>0$

All tan. lines have a positive slope, Increasing

Find $f'(x)$ for $f(x)=x^{-1/2}$.

Discuss domain for both $f(x)$ & $f'(x)$.

$f(x)=\dfrac{1}{x^{1/2}}=\dfrac{1}{\sqrt[2]{x^1}}$

Recall from Algebra

$x^{-n}=\dfrac{1}{x^n}$

$x^{\frac{m}{n}}=\sqrt[n]{x^m}$

$f(x)=\dfrac{1}{\sqrt{x}}$ $\quad f(x)>0$

Domain $(0,\infty)$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\frac{1}{\sqrt{x+h}}-\frac{1}{\sqrt{x}}}{h}$$

LCD $=\sqrt{x}\sqrt{x+h}$

$$=\lim_{h\to 0}\frac{\sqrt{x}-\sqrt{x+h}}{h\sqrt{x}\sqrt{x+h}}$$

$$=\lim_{h\to 0}\frac{(\sqrt{x}-\sqrt{x+h})(\sqrt{x}+\sqrt{x+h})}{h\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

$$=\lim_{h\to 0}\frac{\overbrace{x-(x+h)}^{-\cancel{h}}}{\cancel{h}\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}$$

$$=\lim_{h\to 0}\frac{-1}{\sqrt{x}\sqrt{x+h}(\sqrt{x}+\sqrt{x+h})}=\frac{-1}{\sqrt{x}\sqrt{x}\cdot 2\sqrt{x}}$$

$$=\frac{-1}{2x\sqrt{x}} \qquad \frac{-}{(+)(+)(+)}$$

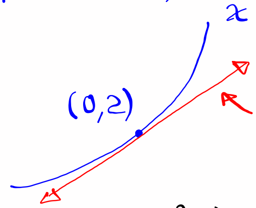
Domain $f'(x)$ $(0,\infty)$

$$=\frac{-\sqrt{x}}{2x^2}$$

$f'(x)<0$

$m_{\text{tan. line}}<0$, All tan. lines have neg. slope. Decreasing

Find eqn of tan. line at $x=0$ to the graph of $f(x)=\dfrac{2}{x+1}$.



$m_{\text{tan. line}} = f'(x)\Big|_{(0,2)} = \dfrac{-2}{(0+1)^2} = \boxed{-2}$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{\dfrac{2}{x+h+1}-\dfrac{2}{x+1}}{h}$$

LCD $=(x+h+1)(x+1)$

$$=\lim_{h\to 0}\frac{2(x+1)-2(x+h+1)}{h(x+h+1)(x+1)}=\lim_{h\to 0}\frac{-2\cancel{h}}{\cancel{h}(x+\cancel{h}^{0}+1)(x+1)}$$

$$=\frac{-2}{(x+1)^2}$$

Point $(0,2)$

$m_{\text{tan. line}}=-2$

$y-y_1=m(x-x_1)$

$y-2=-2(x-0)$ $\rightarrow$ $\boxed{y=-2x+2}$

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

$$f'(a)=\lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$$

Let $x=a+h \rightarrow h=x-a$

as $h\to 0$, $x\to a$

$$\boxed{f'(a)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}}$$

Find the first derivative of $f(x) = \sin x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

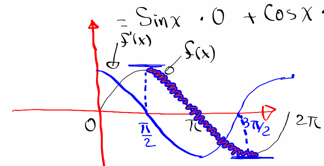
$$= \lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{\sin x\,[\cos h - 1] + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \frac{\sin x\,[\cos h - 1]}{h} + \lim_{h \to 0} \frac{\cos x \sin h}{h}$$

$$= \sin x \lim_{h \to 0} \underbrace{\frac{\cos h - 1}{h}}_{\to 0} + \cos x \lim_{h \to 0} \underbrace{\frac{\sin h}{h}}_{\to 1}$$

$$= \sin x \cdot 0 + \cos x \cdot 1 = \boxed{\cos x}$$



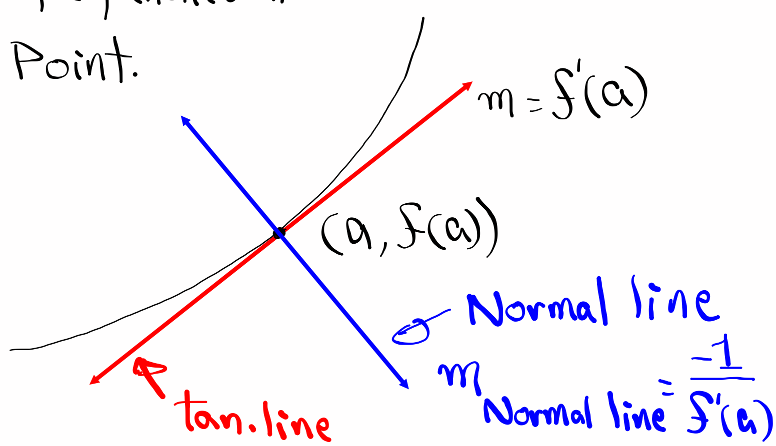
$f'(x) > 0$ $\quad (0, \frac{\pi}{2})$ $\quad f(x)$ increasing

$f'(x) < 0$ $\quad (\frac{\pi}{2}, \frac{3\pi}{2})$ $\quad f(x)$ Decreasing

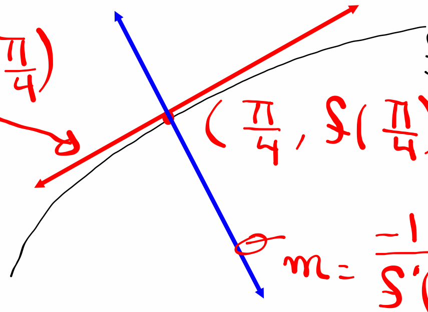
$f'(x) > 0$ $\quad (\frac{3\pi}{2}, 2\pi)$ $\quad f(x)$ increasing

Normal line

It is a line perpendicular to the tan. line at the tan. Point.



Find slope of the normal line to the graph of $f(x) = \sin x$ at $x = \frac{\pi}{4}$.



$m = f'\left(\frac{\pi}{4}\right)$

$f(x) = \sin x$

$\left(\frac{\pi}{4}, f\left(\frac{\pi}{4}\right)\right)$

$m = \dfrac{-1}{f'(\pi/4)}$

$m = \dfrac{-1}{\frac{1}{\sqrt{2}}} = \boxed{-\sqrt{2}}$

Earlier

$f(x) = \sin x$

$f'(x) = \cos x$

$f'\left(\frac{\pi}{4}\right) = \cos\frac{\pi}{4}$

$= \dfrac{1}{\sqrt{2}}$

For $\varepsilon > 0$, find $\delta > 0$ such that

$\lim_{x \to 2} (x^2 + 5x - 3) = 11$

1) Verify the limit ✓

2) $f(x) = x^2 + 5x - 3$, $a = 2$, $L = 11$

3) $|f(x) - L| < \varepsilon$ whenever $|x - a| < \delta$

$|x^2 + 5x - 3 - 11| < \varepsilon$ " $|x - 2| < \delta$

$|x^2 + 5x - 14| < \varepsilon$ " $|x - 2| < \delta$

Keep

$\underbrace{|x+7|}_{\text{Bound}}\,|x-2| < \varepsilon$

we wish $\delta \le 1$

$|x - 2| < \delta \le 1$

$|x - 2| < 1$

$-1 < x - 2 < 1$

Add 9

$-10 < 8 < x + 7 < 10$

$-10 < x + 7 < 10$

$|x + 7| < 10$

$|x+7|\,|x-2| < 10|x-2| < \varepsilon$

$|x - 2| < \dfrac{\varepsilon}{10}$

Pick

$\delta = \min\left\{1, \dfrac{\varepsilon}{10}\right\}$